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Differential Inequalities and Error Bounds

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by

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I cordially thank Thomas A. Bray who has programmed the procedure described in Section 6.

ABSTRACT

A unified theory on a certain type of inequalities for abstract linear operators and, in particular, for ordinary linear differential operators of the first, second, and fourth order is developed. The statements which involve these inequalities lead to a principle of error estimation. With a programmed procedure, approximate solutions and corresponding error bounds have been calculated for some examples of the second and fourth order.

1. INTRODUCTION

This paper is mainly concerned with linear ordinary differential operators L such that

$$L[u](x) \geq 0 \quad (0 \leq x \leq 1) \quad \text{implies} \quad u(x) \geq 0 \quad (0 \leq x \leq 1)$$

for all functions $u(x)$ which possess derivatives of sufficient high order and satisfy certain equations, or inequalities, at the boundary points $x = 0$ and $x = 1$. In particular, we are interested in how this property can be applied in order to obtain error bounds for an approximate solution of a boundary value problem which involves the operator L.

In Section 2, an abstract theory on inverse-positive linear operators M is developed. This theory is then applied to differential equations of the first and second order (Section 3), to systems of such equations (Section 4), and to some differential equations of the fourth order (Section 5). For all of these problems, we obtain in this way principles for error estimation. In Section 6, we finally give some numerical results which have been gained by a programmed procedure based on the error estimation principle just mentioned.

Many results concerning inequalities for ordinary linear differential operators can be found in the literature. In order to prove these results, several different methods have been used. This paper shall show that many of those results, and new results also, can be derived in a rather simple way from simple abstract theorems. In particular, this is true for results which are useful for numerical calculations. Some of the theorems in this paper are already known, for example, the basic Theorem 1 in Section 2, and its applications to differential operators of first and second order, and also the remarks concerning iterative procedures in Section 2.6. Most of the other results seem to be new.

The results in this paper can be generalized. For example, the abstract theorems in Section 2, as well as the numerical procedure in Section 6, can be applied to linear partial differential equations also. Moreover, Theorem 1 in Section 2 is a special case of a theorem on nonlinear inverse-monotonic operators M. This more general theorem can be applied to nonlinear ordinary and partial differential equations, and a similar procedure as that in Section 6 can be used to obtain error bounds for certain types of such nonlinear equations. While a survey on such applications to nonlinear equations will be given elsewhere [8], we restrict ourselves here to linear operators and ordinary differential equations in order to be able to give more details.

2. INVERSE-POSITIVE OPERATORS

Let $R = \{u,v,...\}$ and $S = \{U,V,...\}$ be linear partially ordered spaces, and let M be a linear operator which maps R into S.

Under what conditions is it true that

$$Mu \geq 0 \ \textit{implies} \ u \geq 0$$

for $u \in R$?

A linear operator with this property will be called *inverse-positive,* because the above implication holds if any only if

$$M \ \textit{has a positive inverse,}$$

i.e., M^{-1} exists, and $U \geq 0$ implies $M^{-1}U \geq 0$ for U in the domain of M^{-1}.

2.1 SUFFICIENT CONDITIONS

Let R be Archimedian, i.e.

$$nu \leq v \quad (n = 1,2,...) \ \textit{implies} \ u \leq 0$$

if u,v are arbitrary fixed elements in R. We define a second order relation $u \prec v$ in R (u strongly smaller than v) by:

$u \succ 0$ (u *strongly positive*) *iff* u *has the following*

PROPERTY σ: *For each* $v \in R$ *there exists a natural number* n *such that* $v \leq nu$.

Moreover, we require that there is defined a second order relation $U \prec V$ in S, also. This relation may be defined by Property σ, or may not, but it shall satisfy the following conditions:

$U \succ 0$, $V \geq 0$ imply $U + V \succ 0$; $U \succ 0$, $\lambda > 0$ imply $\lambda U \succ 0$;

$U \succ 0$ implies $U \geq 0$.

These assumptions shall hold throughout this Section 2.

THEOREM 1: *Let the following assumptions be satisfied.*

Assumption I: For $w \in R$,

$$w \geq 0, \quad Mw \succ 0 \quad \textit{imply} \quad w \succ 0.$$

Assumption II: There exists $z \in R$ *such that*

$$z \geq 0, \quad Mz \succ 0. \tag{2.1}$$

Then for $u \in R$,

$$Mu \geq 0 \quad \textit{implies} \quad u \geq 0.$$

PROOF (see Figure 1): Suppose that $Mu \geq 0$, but not $u \geq 0$. Then, because R is Archimedian, there exists a smallest number $\lambda_0 > 0$ such that $u + \lambda_0 z \geq 0$. The element $w = u + \lambda_0 z$ satisfies $w \geq 0$ but not $w \succ 0$. On the other hand, Assumption II yields $Mw = Mu + \lambda_0 Mz \succ 0$, and therefore, $w \succ 0$ because of Assumption I. This contradiction shows that the Theorem is true. (Notice, that $z \succ 0$ because of Assumption I. For a more detailed proof, see [7].)

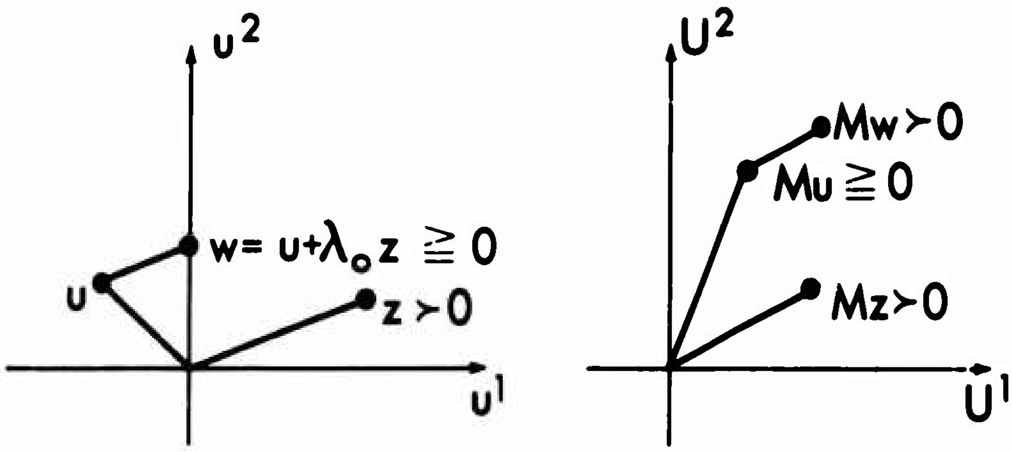

Figure 1: Illustration of the proof with $R = S = R^2$;
$u = (u^1, u^2) \geq 0$ iff $u^1 \geq 0$, $u^2 \geq 0$; $u \succ 0$ iff $u^1 > 0$, $u^2 > 0$;
and the same order relations in S.

SPECIAL CASE 1: If $M = A-B$ with a linear operator A satisfying Assumption I, and a positive linear operator B, then M satisfies Assumption I. In particular, $A = I$ satisfies Assumption I.

2.2 NECESSARY CONDITIONS

We will prove that the sufficient conditions of Theorem 1 are necessary "in general".

THEOREM 2: *Suppose that* M *is inverse-positive. Then the following statements hold.*

1. *The operator* M *satisfies Assumption I if the second order relation in* S *has Property* σ.

2. *The operator* M *satisfies Assumption II if its range contains at least one strongly positive element.*

PROOF: 1) Suppose that $w \geq 0$, $Mw \succ 0$ for some $w \in R$ and let v be any element in R. Then, if the second order relation in S has Property σ, there exists a number n satisfying $Mv \leq nMw$. This inequality yields $v \leq nw$ for inverse-positive M. Because there exists such a number n for every $v \in R$, we have $w \succ 0$ by definition.

2) Let $r \in S$ be any strongly positive element in the range of M and $Mz = r$. Then $Mz \geq 0$ and therefore $z \geq 0$, because M is inverse-positive. Thus, z satisfies Assumption II.

2.3 CONNECTED CLASSES OF OPERATORS

For many important classes of operators, it is easy to prove that Assumption I is satisfied (see, for example, Sections 3 and 4). If M is inverse-positive, the element z occurring in Assumption II may be constructed by solving the equation $Mz = r$ for any $r \succ 0$ (see the second part of the proof of Theorem 2). Considering, however, certain "connected" classes of linear operators M, instead of a single operator, one need prove Assumption II for only one operator of such a class.

Let $\mathfrak{M}$ be a set of linear operators M which map R into S and satisfy Assumption I.

A one parameter family $z(t) \in R$ $(0 \leq t \leq 1)$ will be called *continuous* at $t_0 \in (0,1)$ if for each natural n there exists a positive number $\delta(n)$ such that

$$-\frac{1}{n}e \leq z(t) - z(t_0) \leq \frac{1}{n}e \quad \text{for} \quad |t-t_0| \leq \delta(n), \tag{2.2}$$

where $e \geq 0$ denotes an arbitrary element in R (which may depend on z and t_0).

LEMMA: *Suppose that for each pair of operators $M_0, M_1 \in \mathfrak{M}$ there exists a family of operators $M(t) \in \mathfrak{M}$ and a continuous family $z(t) \in R$ $(0 \leq t \leq 1)$, such that*

$$M(0) = M_0, \quad M(1) = M_1, \quad M(t)z(t) \succ 0 \quad (0 \leq t \leq 1).$$

Let, moreover, at least one operator $\tilde{M} \in \mathfrak{M}$ be inverse-positive.

Then all operators in $\mathfrak{M}$ are inverse-positive.

PROOF: Suppose $M_1 \in \mathfrak{M}$ is not inverse-positive. Define $M_0 = \tilde{M}$ and let $M(t), z(t)$ have the properties mentioned above. The inequality $M_0 z(0) \succ 0$ implies $z(0) \geq 0$ because M_0 is inverse-positive. The two last inequalities yield $z(0) \succ 0$ because of Assumption I. Thus, there exists a smallest positive number $t_0 \leq 1$ such that $z(t) \succ 0$ for $0 \leq t < t_0$. For this number t_0, the right hand inequality in (2.2) yields

$$n(-z(t_0)) \leq e - nz(t) \leq e \qquad \text{if} \qquad 0 \leq t_0 - \delta \leq t < t_0.$$

Therefore, we have $-z(t_0) \leq 0$ because R is Archimedian.

If $t_0 = 1$, the element $z(1)$ satisfies (2.1) for $M = M_1$ and, therefore, M_1 is inverse-positive according to Theorem 1. The inequality $t_0 < 1$, however, contradicts the definition of t_0. Applying Assumption I to $M(t_0)$, we get $z(t_0) \succ 0$. Therefore, there exists a natural number m with the property: $e \leq mz(t_0)$. For $n > m$, the left hand inequality in (2.2) yields

$$0 \prec (1 - \frac{m}{n}) z(t_0) \leq z(t) \qquad \text{if} \qquad |t - t_0| \text{ small enough.}$$

That means, $z(t) \succ 0$ is true for a larger interval than $[0, t_0)$.

One need not always actually construct the elements $z(t)$. In order to prove their existence, some knowledge about the inverse operators of $M \in \mathfrak{M}$ is sufficient.

THEOREM 3: *Let the following assumptions be satisfied.*

1. a) *All operators* $M \in \mathfrak{M}$ *have an inverse.*

   b) *The ranges of all operators* $M \in \mathfrak{M}$ *have at least one element* $r \succ 0$ *in common.*

   c) *For each pair* M_0, M_1 *of operators in* $\mathfrak{M}$ *there exists a family* $M(t) \in \mathfrak{M}$ $(0 \leq t \leq 1)$ *such that* $M(0) = M_0$, $M(1) = M_1$ *and such that the family* $M^{-1}(t)r$ *depends continuously on* t.

2. *At least one operator* $\tilde{M} \in \mathfrak{M}$ *is inverse-positive.*

Then, all operators $M \in \mathfrak{M}$ *are inverse-positive.*

This theorem is an immediate consequence of the lemma, because the elements $z(t) = M^{-1}(t)r$ satisfy all required assumptions.

2.4 SPLIT OPERATORS

If a linear operator M, occurring in some given problem, is not inverse-positive, the problem can often be transformed such that it is described by an inverse-positive operator $\hat{M}$.

Define $\hat{R}$ to be the linear space of ordered pairs $\hat{u} = (u_1,u_2)$ of elements u_1, $u_2 \in R$ with the order relations:

$$\hat{u} \geq 0 \quad \text{iff} \quad u_1 \geq 0, \quad u_2 \leq 0,$$

$$\hat{u} \succ 0 \quad \text{iff} \quad u_1 \succ 0, \quad u_2 \prec 0;$$

and let $\hat{S}$ be similarly defined using ordered pairs of elements of S.

Let M_1, M_2 denote a pair of linear operators mapping R into S such that

$$M = M_1 - M_2, \tag{2.3}$$

and define by

$$\hat{M}\hat{u} = (M_1u_1 - M_2u_2,\ M_1u_2 - M_2u_1) \tag{2.4}$$

an operator $\hat{M}$ which maps $\hat{R}$ into $\hat{S}$.

Then, one may ask whether this operator $\hat{M}$ is inverse-positive, that means, whether for $u_1, u_2 \in R$,

$$\left.\begin{array}{l} M_1u_1 - M_2u_2 \geq 0 \\ M_1u_2 - M_2u_1 \leq 0 \end{array}\right\} \quad \text{imply} \quad \left\{\begin{array}{l} u_1 \geq 0 \\ u_2 \leq 0 \,. \end{array}\right. \tag{2.5}$$

To prove this property, one may apply Theorem 1. *The following assumption is sufficient for Assumption II applied to* $\hat{M}$:

Assumption II': There exists an element $z \in R$ *such that*

$$z \geq 0, \quad (M_1 + M_2)z \succ 0.$$

This follows from the fact that $\hat{z} = (z, -z)$ satisfies Assumption II applied to $\hat{M}$.

SPECIAL CASE 2: Let

$$M = A - B$$

with linear operators A, B, and suppose that A and B can be split into

$$A = A_1 - A_2, \quad B = B_1 - B_2,$$

such that $\hat{A}$ defined by

$$\hat{A}\hat{u} = (A_1u_1 - A_2u_2, \quad A_1u_2 - A_2u_1)$$

satisfies Assumption I, and the operators B_1, B_2 are positive. Then $\hat{M}$ defined by (2.3) with

$$M_1 = A_1 - B_1, \quad M_2 = A_2 - B_2$$

satisfies Assumption I.

For example, if A itself satisfies Assumption I, then $\hat{A}$ constructed with $A_1 = A$, $A_2 = 0$ satisfies Assumption I.

2.5 ERROR ESTIMATION

Let be given an equation

$$Mu = r \tag{2.6}$$

which has a solution $u^* \in R$, and suppose that ϕ is an approximation of u^* with *defect*

$$d[\phi] = -M\phi + r.$$

ERROR ESTIMATION 1: *If* M *is inverse-positive, construct an element* $z \geq 0$ *in* R *and calculate numbers* λ, μ *such that* $-\lambda \leq \mu$ *and*

$$-\lambda Mz \leq d[\phi] \leq \mu Mz. \tag{2.7}$$

Then, the error estimation

$$-\lambda z \leq u^* - \phi \leq \mu z$$

holds.

One proves this statement using $d[\phi] = M(u^* - \phi)$.

REMARK: The element z has to satisfy the inequalities

$$z \geq 0, \quad Mz > 0$$

which are very similar to (2.1) (except in the trivial case $d[\phi] = 0$). Very often, the element z used in the error estimation will satisfy (2.1). In other cases, an element z satisfying Assumption II can be gained by slightly modifying the element z used in the error estimation.

If M is not inverse-positive, one may get an error estimation using the method of splitting of Section 2.4. Let $\hat{M}$ be an operator as described in that section, and let $\hat{M}$ be inverse-positive. Then, the only solution of

$$\hat{M}\hat{u} = \hat{r} \quad \text{with} \quad \hat{r} = (r,r)$$

is $\hat{u}^* = (u^*,u^*)$.

ERROR ESTIMATION 2: *Let* (2.5) *be satisfied for a certain splitting* (2.3). *Construct an element* $z \geq 0$ *in* R *and calculate numbers* λ, μ *such that* $-\lambda \leq \mu$ *and*

$$-\lambda(M_1 + M_2)z \leq d[\phi] \leq \mu(M_1 + M_2)z. \tag{2.8}$$

Then, the following error estimation holds,

$$-\lambda z \leq u^* - \phi \leq \mu z.$$

One proves this statement by applying Error Estimation 1 to $\hat{M}$, $\hat{\phi} = (\phi,\phi)$, $\hat{z} = (z,-z)$.

2.6 ITERATIVE PROCEDURES, EXISTENCE

If M is inverse-positive, the equation $Mu = r$ has at most one solution. Under additional assumptions, one can prove also the existence of such a solution using the means of the preceding sections.

SPECIAL CASE 1: Consider the special case $M = A - B$ at the end of Section 2.1 with A satisfying Assumption I and B positive. Assume, moreover, that M, and therefore also A, satisfies Assumption II and let $AR = S$. Then, A is inverse-positive, and the operator T defined by $Tu = A^{-1}r + A^{-1}Bu$ is an isotonic mapping of R into R.

Suppose, that the inequalities (2.7) in Error Estimation 1 are satisfied and define sequences $\{x_{(n)}\}$, $\{y_{(n)}\}$ by

$$x_{(0)} = \phi - \lambda z, \quad y_{(0)} = \phi + \mu z$$

$$Ax_{(n+1)} = r + Bx_{(n)}, \quad Ay_{(n+1)} = r + By_{(n)} \qquad (n = 0,1,2,\ldots),$$

i.e.

$$x_{(n+1)} = Tx_{(n)}, \quad y_{(n+1)} = Ty_{(n)} \qquad (n = 0,1,2,\ldots). \tag{2.9}$$

Then, $x_{(0)} \leq y_{(0)}$, and by applying A^{-1} to (2.7), one gets $x_{(0)} \leq x_{(1)}$, $y_{(1)} \leq y_{(0)}$. Because T is isotonic, these inequalities yield

$$x_{(0)} \leq x_{(1)} \leq \cdots \leq x_{(n)} \leq \cdots \leq y_{(n)} \leq \cdots \leq y_{(1)} \leq y_{(0)}. \quad (2.10)$$

In many cases, one can conclude from (2.9) and (2.10) that the sequences $\{x_{(n)}\}$, $\{y_{(n)}\}$ converge to fixed points $x^* = Tx^*$ and $y^* = Ty^*$, respectively. Each fixed point is a solution of the given equation $Mu = r$. Because M is inverse-positive, $x^* = y^*$. (If only A, instead of M, satisfies Assumption II, we cannot conclude $x^* = y^*$ which, of course, is not necessary to prove the existence of a solution.)

Under suitable additional assumptions on the space R *and the operators* A,B, *the inequalities* (2.7) *of Error Estimation* 1 *yield the existence of a solution* u^* *of the given equation* $Mu = r$.

SPECIAL CASE 2: Let now M be as in the Special Case 2 at the end of Section 2.4. Suppose, moreover, that M satisfies Assumption II' also, and let $\hat{A}\hat{R} = \hat{S}$. (The last relation holds, for example, if $A = A_1$ and $AR = S$.) If then the inequalities (2.8) of Error Estimation 2 are satisfied, let $\{x_{(n)}\}$, $\{y_{(n)}\}$ be the sequences defined by

$$x_{(0)} = \phi - \lambda z, \quad y_{(0)} = \phi + \mu z,$$

$$A_1 x_{(n+1)} - A_2 y_{(n+1)} = r + B_1 x_{(n)} - B_2 y_{(n)},$$

$$A_1 y_{(n+1)} - A_2 x_{(n+1)} = r + B_1 y_{(n)} - B_2 x_{(n)} \qquad (n = 0,1,2,\ldots).$$

These sequences again satisfy (2.10), and, therefore, one very often can conclude that these sequences converge to elements x^*, y^* with

$$A_1x^* - A_2y^* = r + B_1x^* - B_2y^*,$$

$$A_1y^* - A_2x^* = r + B_1y^* - B_2x^*.$$

The last relations are equivalent to $\hat{M}\hat{x}^* = \hat{r}$ with $\hat{x}^* = (x^*, y^*)$, as well as to $\hat{M}\hat{y}^* = \hat{r}$ with $\hat{y}^* = (y^*, x^*)$. Because $\hat{M}$ is inverse-positive, $\hat{x}^* = \hat{y}^*$, i.e. $x^* = y^*$. Consequently, $u^* = x^* = y^*$ is a solution of the given equation. (If only $A = A_1 - A_2$, instead of $M = M_1 - M_2$, satisfies Assumption II', one cannot conclude $x^* = y^*$ but still can prove the existence of a solution, namely, $u^* = \frac{1}{2}(x^* + y^*)$.)

Under suitable additional assumptions on the space R *and the operators* A_1, A_2, B_1, B_2, *the inequalities* (2.8) *of Error Estimation 2 yield the existence of solution* u^* *of the given equation* $Mu = r$.

3. PROBLEMS WITH A SINGLE DIFFERENTIAL EQUATION OF AT MOST THE SECOND ORDER

3.1 SUFFICIENT CONDITIONS FOR INVERSE-POSITIVITY

Consider the boundary value problem

$$\left.\begin{array}{l} L[u] = L[u](x) = a(x)u'' + b(x)u' + c(x)u = s(x) \quad (0 < x < 1), \\ V[u] = \alpha u'(0) + \beta u(0) = A, \quad W[u] = \gamma u'(1) + \delta u(1) = B. \end{array}\right\} \quad (3.1)$$

with given finite real functions $a, b, c,$ and s. We ask for solutions $u \in R$ where R is a linear subset of functions in $C[0,1]$ which have sufficient differentiability properties. At least, we will require that $u \in R$ possesses all derivatives which actually occur in the given problem. (For example, $u''(x)$ shall exist if $a(x) \neq 0$ for that point x.)

Under what conditions do the inequalities

$$\left.\begin{array}{ll} L[u](x) \geq 0 & (0 < x < 1), \\ V[u] \geq 0, & W[u] \geq 0 \end{array}\right\} \textit{imply} \quad u(x) \geq 0 \quad (0 \leq x \leq 1) \tag{3.2}$$

for $u \in R$?

SPECIAL CASE A: The *initial value problem*

$$\left.\begin{array}{ll} L[u] = u' + c(x)u = s(x) & (0 < x \leq 1), \\ V[u] = u(0) = A & \end{array}\right\} \tag{3.3}$$

can be written in the form (3.1) with

$$W[u] = L[u](1), \quad B = s(1).$$

SPECIAL CASE B: The *singular boundary value problem*

$$L[u](x) = a(x)u'' + b(x)u' + c(x)u = s(x) \qquad (0 \leq x \leq 1)$$

with

$$a(0) = a(1) = 0,$$

but without given boundary conditions, can be written in the form (3.1) by defining

$$V[u] = L[u](0), \quad A = s(0); \qquad W[u] = L[u](1), \quad B = s(1).$$

THEOREM 4: *If*

$$a(x) \le 0 \qquad (0 < x < 1), \tag{3.4}$$

$$\alpha \le 0, \qquad \gamma \ge 0 \tag{3.5}$$

and if there exists a function $z \in R$ *such that*

$$\left.\begin{array}{l} z(x) \ge 0 \quad (0 \le x \le 1), \\ L[z](x) > 0 \quad (0 \le x \le 1), \quad V[z] > 0, \quad W[z] > 0, \end{array}\right\} \tag{3.6}$$

then, (3.2) *holds for* $u \in R$.

PROOF: Let S be the space of vectors $U = (s(x), A, B)$ with real functions $s(x)$, defined on $(0,1)$, and constants A, B. Then, the operator $Mu = (L[u](x), V[u], W[u])$ maps R into S.

Define, moreover,

$$\begin{array}{llll} u \ge 0 & \text{iff} & u(x) \ge 0 & (0 \le x \le 1); \\ u \succ 0 & \text{iff} & u(x) > 0 & (0 \le x \le 1); \\ U \ge 0 & \text{iff} & s(x) \ge 0 & (0 < x < 1), \quad A \ge 0, \; B \ge 0; \\ U \succ 0 & \text{iff} & s(x) > 0 & (0 < x < 1), \quad A > 0, \; B > 0. \end{array}$$

Then, (3.2) holds iff M is inverse-positive, and it suffices to prove that M satisfies the Assumptions I and II of Theorem 1. To prove Assumption I, let us suppose that $w \ge 0$, $Mw \succ 0$, but not $w \succ 0$. Then, $w(x)$ assumes the minimum $w(x_0) = 0$ at some point $x_0 \in [0,1]$. If $x_0 \in (0,1)$, then $w'(x_0) = 0$, $w''(x_0) \ge 0$, and therefore, $L[w](x_0) \le 0$. If $x_0 = 0$, then $w'(0) \ge 0$, and therefore, $V[w] \le 0$. If $x_0 = 1$, then $w'(1) \le 0$, and therefore, $W[w] \le 0$.

Thus, in each case one gets a contradiction to $Mw \succ 0$.

For the SPECIAL CASE A, the assumptions (3.4), (3.5) are always satisfied. Assumption II is equivalent to (3.6).

For the SPECIAL CASE B, the assumption (3.5) is equivalent to

$$b(0) \leq 0, \quad b(1) \geq 0.$$

For example, these inequalities hold for

$$L[u] = -x(1-x)u'' + \sigma(x-\tfrac{1}{2})u' + c(x)u,$$

if σ is a non-negative constant. If, however, the constant σ is negative, one needs additional boundary conditions in order to form an inverse-positive operator M.

EXAMPLES of functions z: The following functions z satisfy the inequalities (3.6) if the corresponding conditions, stated below, hold.

1) $\quad z(x) \equiv 1,$

if

$$c(x) > 0 \quad (0 < x < 1), \quad \beta > 0, \quad \delta > 0; \tag{3.7}$$

2) $\quad z(x) = \cos(\pi-\varepsilon)(x-\tfrac{1}{2})$

with small enough $\varepsilon > 0$, if

$$\left.\begin{array}{l} L[u] = -u'' + c(x)u \quad \text{with} \quad c(x) > -\pi^2 \quad (0 < x < 1), \\ V[u] = u(0), \quad W[u] = u(1); \end{array}\right\} \tag{3.8}$$

3) $\qquad z(x) = e^{Nx}$

in case of the initial value problem (3.3) with $c(x) > -N \quad (0 \leq x \leq 1)$.

Further functions z which can be used under weaker conditions as, for example, (3.7) are given in [6].

3.2 CONNECTED CLASSES OF EQUATIONS

In this Section 3.2, we consider only operators L with coefficients $a(x)$, $b(x)$, $c(x)$ which are defined and continuous on $[0,1]$. Moreover, we restrict ourselves to boundary value problems (3.1) with

$$a(x) \equiv -1 \quad (0 \leq x \leq 1), \qquad \alpha \leq 0, \qquad \gamma \geq 0, \tag{3.9}$$

and we define $R = C_1[0,1] \cap C_2(0,1)$.

Each triple $M = (L, V, W)$ is then described by its coefficient vector

$$\mathfrak{k} = (b(x), c(x), \alpha, \beta, \gamma, \delta).$$

Let $\mathfrak{M}$ *be a set of such triples* $M = (L,V,W)$ *such that the corresponding coefficient vectors* $\mathfrak{k}$ *form a connected set* $\mathfrak{K}$, i.e. for each pair $\mathfrak{k}_0, \mathfrak{k}_1 \in \mathfrak{K}$, there exists a (component-wise) continuous curve $\mathfrak{k}(t) \in \mathfrak{K} (0 \leq t \leq 1)$ with $\mathfrak{k}_0 = \mathfrak{k}(0)$, $\mathfrak{k}_1 = \mathfrak{k}(1)$. For example, $\mathfrak{M}$ may be a convex set.

THEOREM 5: *Let the following assumptions be satisfied.*

1) For each $M = (L,V,W) \in \mathfrak{M}$, *the homogeneous problem, corresponding to* (3.1), *has no nontrivial solution.*

2) There is at least one $M = (L,V,W) \in \mathfrak{M}$ *such that* (3.8) *is satisfied.*

Then, the implication (3.2) *is true for all* $M = (L,V,W) \in \mathfrak{M}$.

PROOF: The statement of this theorem can be derived from Theorem 3 by considering each triple $M = (L,V,W)$ as an operator defined as in the proof of Theorem 4, and using the following facts.

Because of (3.9) each $M \in \mathfrak{M}$ satisfies Assumption I according to the proof of Theorem 4. Because of Assumption 1, all operators $M \in \mathfrak{M}$ have an inverse M^{-1} which can be applied to $r = (s(x), A,B)$ with $s(x) = 1$, $A = 1$, $B = 1$.

Moreover, for any family of operators $M(t) \in \mathfrak{M} \, (0 \le t \le 1)$ with continuous coefficient vectors $\mathfrak{k}(t)$, the functions $z(t) = M^{-1}(t)r$ depend continuously on t.

Finally, at least one $M \in \mathfrak{M}$ is inverse-positive, because (3.8) holds for at least one $M \in \mathfrak{M}$, and for this operator M Assumption II is satisfied also.

Roughly spoken, Theorem 5 says: If one starts with a triple (L,V,W) such that (3.2) holds, and then changes the coefficients continuously, the implication (3.2) remains true as long as one does not hit a triple (L,V,W) such that the corresponding problem (3.1) does not have a unique solution.

A very simple EXAMPLE is the set $\mathfrak{M}$ of triples $M = (L,V,W)$ with (3.8).

3.3 ERROR ESTIMATION

Consider a problem (3.1) such that (3.2) holds. Suppose that there exists a solution $u^* \in R$, and let $\phi \in R$ be an approximate solution. Then, one can get a bound for the error $u^* - \phi$ in the following way.

ERROR ESTIMATION: *Construct a function* $z \in R$ *and calculate a constant* λ *such that*

$$|-L[\phi](x) + s(x)| \leq \lambda L[z](x) \qquad (0 < x < 1),$$
$$|-V[\phi] + A| \leq \lambda V[z]$$
$$|-W[\phi] + B| \leq \lambda W[z].$$

Then, the error estimation

$$|u^*(x) - \phi(x)| \leq \lambda z(x) \qquad (0 \leq x \leq 1) \tag{3.10}$$

holds.

3.4 A DIFFERENT APPROACH

For ordinary differential equations, one often will choose an approximate solution ϕ which satisfies the given boundary conditions. Then, one does not need (3.2) but only the weaker property:

$$\left.\begin{array}{ll} L[u](x) \geq 0 & (0 \leq x \leq 1) \\ V[u] = W[u] = 0 & \end{array}\right\} \quad \textit{imply} \quad u(x) \geq 0 \quad (0 \leq x \leq 1) \quad (3.11)$$

for $u \varepsilon R$. (We now suppose that the coefficients $a(x)$, $b(x)$, $c(x)$, and $s(x)$ are defined on $[0,1]$.)

THEOREM 6: *If*

$$a(x) \leq 0 \quad (0 \leq x \leq 1),$$

and if there exists a function $z \varepsilon R$ *such that*

$$z(x) \geq 0 \quad (0 \leq x \leq 1),$$

$$L[z](x) > 0 \quad (0 \leq x \leq 1), \quad V[z] = W[z] = 0,$$

then (3.11) *holds for* $u \varepsilon R$.

This theorem can be proved in a similar way as Theorem 4. One now defines $Mu = L[u]$ on the space $\tilde{R}$ of functions $u \varepsilon R$, which satisfy the homogeneous boundary conditions, and then applies Theorem 1 (replacing there R by $\tilde{R}$).

Notice, that for (3.11), *the conditions* (3.5) *are not required.*

Using this different approach, one can consider CONNECTED CLASSES of equations, similarly as in Section 3.2, and one can also derive a corresponding ERROR ESTIMATION:

$$|-L[\phi](x) + s(x)| \leq \lambda L[z](x) \quad (0 \leq x \leq 1)$$

$$\textit{implies} \quad |u^*(x) - \phi(x)| \leq \lambda z(x) \quad (0 \leq x \leq 1),$$

provided $\phi \varepsilon R$ satisfies the given (inhomogeneous) boundary conditions and $z \varepsilon R$ satisfies the corresponding homogeneous boundary conditions.

3.5 ESTIMATION OF THE DERIVATIVE OF A SOLUTION

It is often possible to calculate bounds for the derivative of the solution of a problem (3.1). For illustration, we consider the special case

$$L[u](x) = -u'' + b(x)u' + c(x)u = s(x) \qquad (0 \leq x \leq 1)$$

$$u(0) = A, \quad u(1) = B.$$

(A more general discussion will be given elsewhere.)

Suppose there exists a solution $u^* \in R = C_1[0,1] \cap C_2(0,1)$ and there has been gained an error estimation

$$|u^*(x) - \phi(x)| \leq \lambda z(x) \qquad (0 \leq x \leq 1) \tag{3.12}$$

for an approximation $\phi \in R$ for u^*. Let $z \in R$ satisfy the homogeneous boundary conditions $z(0) = z(1) = 0$. Let, moreover, $b(x)$ be bounded: $|b(x)| \leq N$ $(0 \leq x \leq 1)$. Then, one can calculate a bound for $(u^* - \phi)'$.

ESTIMATION OF THE DERIVATIVE: *Construct a function* $\zeta \in C_1[0,1]$ *and calculate a constant* $\mu \geq 0$ *such that*

$$|-L[\phi](x) + s(x)| \leq \mu[\zeta'(x) - b(x)\zeta(x)] - \lambda|c(x)|z(x) \quad (0 \leq x \leq 1), \tag{3.13}$$

$$\lambda z'(0) \leq \mu\zeta(0). \tag{3.14}$$

Then,

$$|u^{*\prime}(x) - \phi'(x)| \leq \mu\zeta(x) \qquad (0 \leq x \leq 1).$$

PROOF: Because of (3.12), (3.13), (3.14), we get the following estimations for $v^* = u^{*\prime}$:

$$(v^* - \phi')' - b(v^* - \phi') = L[\phi] - s + c(u^* - \phi)$$
$$\leq \mu(\zeta' - b\zeta) \qquad (0 \leq x \leq 1),$$
$$(v^* - \phi')(0) = \lim_{h \to +0} h^{-1}(u^*(h) - \phi(h))$$
$$\leq \lambda \lim_{h \to +0} h^{-1} z(h) = \lambda z'(0) \leq \mu\zeta(0)$$

According to Theorem 4, the implication (3.2) holds for the operators

$$L'[v] = v' - bv, \quad \text{defined on} \quad (0,1),$$
$$V'[v] = v(0), \quad W'[v] = L'[v](1)$$

(see Special Case A and notice that Assumption II is satisfied with $z = e^{Nx}$.) Therefore, the inequalities proved above yield $v^* - \phi' \leq \mu\zeta$ $(0 \leq x \leq 1)$. In a similar way, one shows that $-\mu\zeta \leq v^* - \phi'$ $(0 \leq x \leq 1)$ which then proves the error estimation.

REMARK: *The conditions* (3.13), (3.14) *may be replaced by similar conditions which contain an inequality at* $x = 1$, *instead of* (3.14).

4. SYSTEMS OF EQUATIONS OF AT MOST THE SECOND ORDER

4.1 SUFFICIENT CONDITIONS FOR INVERSE-POSITIVITY

We consider now a boundary value problem for m unknown functions $u_1(x),\ldots,u_m(x)$:

$$L_i[u] = a_i(x)u_i'' + b_i(x)u_i' + \sum_{k=1}^{m} c_{ik}(x)u_k = s_i(x) \quad (0 < x < 1),$$

$$V_i[u] = \alpha_i u_i'(0) + \sum_{k=i}^{m} \beta_{ik}u_k(0) = A_i,$$

$$W_i[u] = \gamma_i u_i'(1) + \sum_{k=1}^{m} \delta_{ik}u_k(1) = B_i \qquad (i = 1,2,\ldots,m).$$

Let R be a linear set of m-tuples $u = (u_1(x), u_2(x),\ldots,u_m(x))$ such that all $u_k \in C[0,1]$ and such that for $u \in R$, the components $u_k(x)$ possess all derivatives which actually occur in the given problem. We ask for solutions $u \in R$.

Again, this problem includes *initial value problems* and *singular boundary value problems* as special cases.

THEOREM 4': *Suppose that*

$$a_i(x) \leq 0 \quad (0 < x < 1), \quad \alpha_i \leq 0, \quad \gamma_i \geq 0 \tag{4.1}$$

and

$$c_{ik}(x) \leq 0 \quad (0 < x < 1), \quad \beta_{ik} \leq 0, \quad \delta_{ik} \geq 0 \quad \text{for } i \neq k \tag{4.2}$$

$(i,k = 1,2,\ldots,m)$.

Moreover, let there exist $z = (z_1,\ldots,z_m) \in R$ *such that*

$z_i(x) \geq 0 \quad (0 \leq x \leq 1)$,

$L_i[z](x) > 0 \quad (0 < x < 1), \quad V_i[z] > 0, \quad W_i[z] > 0, \quad (i = 1,2,\ldots,m).$

Then, the inequalities

$$L_i[u](x) \geq 0 \quad (0 \leq x \leq 1)$$

$$V_i[u] \geq 0, \quad W_i[u] \geq 0 \qquad (i = 1,2,\ldots,m)$$

imply

$$u_i(x) \geq 0 \quad (0 \leq x \leq 1; \quad i = 1,2,\ldots,m).$$

This theorem can be proved by generalizing the proof of Theorem 4. One defines M to be the operator

$$\begin{aligned} Mu &= (L_1[u],\ldots,L_m[u]; \; V_1[u],\ldots,V_m[u]; \; W_1[u],\ldots,W_m[u]) \\ &= (L_i[u], \; V_i[u], \; W_i[u]), \end{aligned} \tag{4.3}$$

which maps R into the space S of elements $U = (s_i(x), A_i, B_i)$ $(i = 1,2,\ldots,m)$ with functions $s_i(x)$ defined on $(0,1)$ and constants A_i, B_i. The inequalities $u \geq 0$, $U \geq 0$ shall be understood as holding component-and pointwise.

4.2 SPLITTING OF THE PROBLEM

If the assumptions of Theorem 4' are satisfied, one can get an error estimation for an approximate solution ϕ. In many cases, however, an error estimation is also possible, if (4.2) does not hold. We apply the method of splitting, described in Section 2.4, to the operator M in (4.3). Let

$$L_{i1}[u] = a_i u_i'' + b_i u_i' + c_{ii} u_i - \Sigma' c_{ik}^- u_k,$$

$$V_{i1}[u] = \alpha_i u_i'(0) + \beta_{ii} u_i(0) - \Sigma' \beta_{ik}^- u_k(0),$$

$$W_{i1}[u] = \gamma_i u_i'(1) + \delta_{ii} u_i(1) - \Sigma' \delta_{ik}^{-} u_k(1),$$

$$L_{i2} = L_{i1} - L, \quad V_{i2} = V_{i1} - V, \quad W_{i2} = W_{i1} - W \qquad (i = 1,2,\ldots,m)$$

where we have used the notations

$$\Sigma' = \sum_{\substack{k=1 \\ k \neq i}}^{m}, \quad f^{-} = \tfrac{1}{2}(|f| - f).$$

Define then, similar as in (4.3),

$$M_1 u = (L_{i1}[u],\ V_{i1}[u],\ W_{i1}[u]),\ M_2 u = (L_{i2}[u],\ V_{i2}[u],\ W_{i2}[u]).$$

Then, for the operator $\hat{M}$, corresponding to the splitting $M = M_1 - M_2$ (see (2.4)), Assumption I is always satisfied if only (4.1) holds.

The corresponding operator $\tilde{M} = M_1 + M_2$ is

$$\tilde{M}u = (\tilde{L}_i[u],\ \tilde{V}_i[u],\ \tilde{W}_i[u])$$

with

$$\tilde{L}_i[u] = a_i u_i'' + b_i u_i' + c_{ii} u_i - \Sigma' |c_{ik}| u_k,$$

$$\tilde{V}_i[u] = \alpha_i u_i'(0) + \beta_{ii} u_i(0) - \Sigma' |\beta_{ik}| u_k(0),$$

$$\tilde{W}_i[u] = \gamma_i u_i'(2) + \delta_{ii} u_i(1) - \Sigma' |\delta_{ik}| u_k(1).$$

With these notations, Assumption II' is as follows:

Assumption II': Let there exist $z = (z_1,\ldots,z_m) \in R$ *such that*

$$z_i(x) \geq 0 \quad (0 \leq x \leq 1),$$

$$\tilde{L}_i[z](x) > 0 \quad (0 < x < 1), \quad \tilde{V}_i[z] > 0, \quad \tilde{W}_i[z] > 0 \quad (i = 1,2,\ldots,m).$$

4.3 ERROR ESTIMATION

Let the given problem in Section 4.1 have a solution $u^* \in R$, and let $\phi \in R$ denote an approximation for u^*. Suppose, that the inequalities (4.1) hold and that Assumption II' in Section 4.2 is satisfied. Then, one can get a bound for the error $u^* - \phi$ in the following way.

ERROR ESTIMATION: *Construct an element* $z \in R$ *and calculate a constant* λ *such that*

$$|-L_i[\phi](x) + s_i(x)| \leq \lambda \tilde{L}_i[z](x) \quad (0 < x < 1),$$

$$|-V_i[\phi] + A_i| \leq \lambda \tilde{V}_i[z],$$

$$|-W_i[\phi] + B_i| \leq \lambda \tilde{W}_i[z] \qquad (i = 1,2,\ldots,m).$$

Then,

$$|u^{*i}(x) - \phi^i(x)| \leq \lambda z_i(x) \quad (0 \leq x \leq 1; \quad i = 1,2,\ldots,m).$$

For z *in Assumption II', such a constant* λ *always exists.*

This statement is a consequence of Error Estimation 2 in Section 2.5.

5 BOUNDARY VALUE PROBLEMS OF THE FOURTH ORDER

5.1 THE GENERAL PROBLEM

Consider a differential equation

$$L[u](x) = s(x) \quad (0 \leq x \leq 1),$$

together with two boundary conditions at $x = 0$:

$$V_i[u] = A_i \quad (i = 1,2),$$

and two such conditions at $x = 1$:

$$W_i[u] = B_i \quad (i = 1,2).$$

Let L be a linear differential operator of the fourth order such that the coefficient of u^{IV} is positive in $[0,1]$, and let $V_i[u]$, $W_i[u]$ be linear combinations of derivatives of u up to the third order. For simplicity, we assume that the coefficient-functions of L are analytic on $[0,1]$.

Under what conditions do the relations

$$\left.\begin{array}{l} L[u](x) \geq 0 \quad (0 \leq x \leq 1) \\ V_i[u] = W_i[u] = 0 \quad (i = 1,2) \end{array}\right\} \quad \textit{imply} \quad u(x) \geq 0 \quad (0 \leq x \leq 1) \qquad (5.1)$$

for all analytic functions u *on* $[0,1]$?

This is true iff the corresponding Green's function exists and is non-negative. Therefore, if this implication is true for all analytic functions u, it is also true for all $u \in C_4[0,1]$.

There are some difficulties in applying Theorem 1 to this problem directly. In applying Theorem 1 to second order problems, we used the fact that the first and second derivatives of a function have to satisfy certain necessary conditions at a point where this function assumes an extremum. However, at such a point nothing can be said about higher derivatives which occur in problems of higher order. Therefore, we split the given operator L.

We construct linear differential operators of the second order L_1, L_2 such that for all analytic functions u:

$$L_2[v] = w(x)(L[u] + q(x)u) \quad (0 \leq x \leq 1) \tag{5.2}$$

with

$$L_1[u] = v \tag{5.3}$$

and some fixed (analytic) functions w, q.

Then, we ask for conditions such that for all analytic $u(x)$:

$$\left.\begin{array}{l} L[u](x) \geq 0 \quad (0 \leq x \leq 1), \\ V_i[u] = W_i[u] = 0 \quad (i = 1,2) \end{array}\right\} \text{ imply } \left\{\begin{array}{l} u(x) \geq 0, \\ v(x) \geq 0 \ (0 \leq x \leq 1). \end{array}\right. \tag{5.4}$$

In order to gain such conditions, we formulate this problem in abstract terms and then apply Theorem 1.

Let R be the space of all functions u which are analytic on $[0,1]$ and satisfy the boundary conditions $V_i[u] = W_i[u] = 0$ $(i = 1,2)$, and let S be the set of analytic functions U on $[0,1]$. Define for $u \in R$, respectively $U \in S$:

$$u \geq 0 \quad \text{iff} \quad \left\{\begin{array}{l} u(x) \geq 0, \\ v(x) \geq 0 \quad (0 \leq x \leq 1) \end{array}\right. \tag{5.5}$$

with (5.3),

$$U \geq 0 \quad \text{iff} \quad U(x) \geq 0 \quad (0 \leq x \leq 1),$$

and let $u \succ 0$, $U \succ 0$ denote the corresponding strong order relations, defined by Property σ. Then, $Mu = L[u]$ is a mapping of R into S, and (5.4) holds iff M is inverse-positive.

Thus, we may apply Theorem 1 in order to get sufficient conditions for the property (5.4). It is not possible to present here a general theory of this kind. A first paper on this topic will soon be published elsewhere [9]. Instead of reviewing general results, we will discuss here a special, but typical case.

5.2 A SPECIAL PROBLEM

For the special boundary value problem

$$L[u](x) = u^{IV} - bu'' + c(x)u = s(x) \quad (0 \leq x \leq 1), \tag{5.6}$$

$$u(0) = A_1, \quad u'(0) = A_2, \quad u(1) = B_1, \quad u'(1) = B_2 \tag{5.7}$$

with constant b, let us choose the following splitting:

$$\begin{aligned} L_1[u] &= -pu'' + p'u' + (P + \gamma)u, \\ L_2[v] &= -pv'' + p'v' + (P - \gamma)v \end{aligned} \tag{5.8}$$

where $p(x)$ denotes an analytic function, $P = -\frac{1}{2}p'' + \frac{1}{2}bp$, and γ is a constant. For these operators, (5.2) (5.3) hold with

$$w = p^2, \quad q \equiv L_2[P + \gamma] - cp^2.$$

In particular, we choose

$$p(x) = \int_0^x g(\xi)d\xi \tag{5.9}$$

where g denotes a function, such that for some constant c_0:

$$g^{IV} - 2bg'' + b^2 g = 4c_0 g \quad (0 \leq x \leq 1) \tag{5.10}$$

$$\left.\begin{array}{l} g(0) = 0, \quad g'(0) = 1, \quad g(x) = -g(1-x) \quad (0 \le x \le 1), \\ g(x) > 0 \quad (0 < x < \tfrac{1}{2}). \end{array}\right\} \tag{5.11}$$

Moreover, let

$$\gamma = -\tfrac{1}{2}(1-\varepsilon) \tag{5.12}$$

with some fixed ε such that $0 < \varepsilon < 2$.

THEOREM 6: *Suppose the following assumptions are satisfied.*

I. For some constant c_0 *with*

$$c(x) \le c_0 \quad (0 \le x \le 1), \tag{5.13}$$

there exists a solution g *of the equation* (5.10) *satisfying* (5.11). *Let then* L_1, L_2 *denote the operators described above.*

II. There exists a function $z \in R$ *such that*

$$\left.\begin{array}{r} L[z](x) > 0 \quad (0 \le x \le 1), \\ z(x) \ge 0, \quad L_1[z](x) \ge 0 \quad (0 \le x \le 1). \end{array}\right\} \tag{5.14}$$

Then, for all analytic functions u *on* $[0,1]$, *the relations*

$$\left.\begin{array}{l} L[u] = u^{IV} - bu'' + c(x)u \ge 0 \quad (0 \le x \le 1), \\ u(0) = u'(0) = u(1) = u'(1) = 0 \end{array}\right\}$$

$$\text{imply} \left\{\begin{array}{l} u(x) \ge 0, \\ L_1[u](x) \ge 0 \quad (0 \le x \le 1) \end{array}\right. \tag{5.15}$$

PROOF: We will show that the operator M defined in Section 5.1 satisfies the assumptions of Theorem 1, for the special case which we consider here.

The functions $v = L_1[u]$ with $u \in R$ satisfy $v(0) = v'(0) = v(1) = v'(1) = 0$. There are, however, functions v of that kind such that $v''(0) \neq 0$, $v''(1) \neq 0$. Therefore, and because $u \in R$ satisfies the homogeneous boundary conditions, corresponding to (5.7), for $u \in R$:

$$u \succ 0 \quad \text{iff} \quad \begin{cases} u(x) > 0, \quad v(x) > 0 \quad (0 < x < 1), \\ u''(0) > 0, \quad u''(1) > 0, \quad v''(0) > 0, \quad v''(1) > 0. \end{cases} \tag{5.16}$$

Moreover, for $U \in R$:

$$U \succ 0 \quad \text{iff} \quad U(x) > 0 \quad (0 \leq x \leq 1).$$

Thus, Assumption II of Theorem 1 is satisfied because of the conditions (5.14), and Assumption I of Theorem 1 is equivalent to the following condition: For $u \in R$, the inequalities

$$\begin{aligned} &L[u](x) > 0 \quad (0 \leq x \leq 1) \\ &u(x) \geq 0, \quad v(x) = L_1[u](x) \geq 0 \quad (0 \leq x \leq 1) \end{aligned} \tag{5.17}$$

imply the inequalities in (5.16).

The function p in (5.9) is positive in the open interval $(0,1)$ because of the properties of g. The function $q_0 = L_2[P + \gamma] - c_0 p^2$ is constant because

$$2q_0' = p[g^{IV} - 2bg'' + (b^2 - 4c_0)g] \equiv 0 \quad (0 \leq x \leq 1).$$

The constant value of q_0 is

$$q_0(x) = q_0(0) = \tfrac{1}{4}[1 - (1-\varepsilon)^2] > 0.$$

Therefore, and because of (5.13),

$$q(x) = q_0(x) + (c_0 - c(x))p^2(x) \geq 0 \quad (0 \leq x \leq 1).$$

Let now the inequalities (5.17) be satisfied for some $u \varepsilon R$. Then, the function on the right-hand side of the equality (5.2) is positive in $(0,1)$, and therefore, one can prove, by the same method that was used for second order equations, that $v(x) > 0 \quad (0 < x < 1)$. But then, the same proof applied to equation (5.3) yields $u(x) > 0 \quad (0 < x < 1)$.

In order to prove $u''(0) > 0$ and $v''(0) > 0$, let us assume that $v''(0) > 0$ is not satisfied. Then, $v''(0) = 0$ where

$$v''(0) = (+ \tfrac{1}{2}p''(0) + \gamma)u''(0) = \tfrac{1}{2} \varepsilon u''(0), \tag{5.18}$$

and therefore, $u''(0) = 0$. These equalities then yield

$$0 \leq v'''(0) = (-\frac{1}{2}p''(0) + \gamma)u'''(0) = (-1 + \frac{\varepsilon}{2})u'''(0) \quad \text{and} \quad u'''(0) \geq 0$$

which is only possible if $v'''(0) = u'''(0) = 0$. In this case, however,

$$0 \leq v^{IV}(0) = (-\frac{5}{2}p''(0) + \gamma)u^{IV}(0) = (-3 + \frac{\varepsilon}{2})L[u](0)$$

which contradicts (5.17). Therefore, $v''(0) > 0$ and also $u''(0) > 0$ because of (5.18).

In the same way, one proves $u''(1) > 0$, $v''(1) > 0$.

EXAMPLE 1: For

$$4c_0 = (4\pi^2 + b)^2,$$

the problem (5.10), (5.11) has the solution

$$g(x) = -\frac{1}{2\pi} \sin 2\pi\, (x - \tfrac{1}{2}).$$

EXAMPLE 2: For $c_0 = b = 0$, the problem (5.10), (5.11) has the solution $g(x) = x(1-x)(1-2x)$, so that

$$p(x) = \tfrac{1}{2}x^2(1-x)^2.$$

Moreover, the function $z = x^2(1-x)^2$ satisfies (5.14) if $c(x) \equiv 0$.

5.3 CONNECTED CLASSES OF PROBLEMS

Theorem 3 cannot be applied immediately because the order relations defined in R now depend on the operator M. However, one still can apply the basic idea of that theorem.

Again, we consider the problem (5.6), (5.7); but first we assume that $c(x) \equiv c_0 =$ const.

Let $\mathfrak{K}$ *be a connected set of coefficient vectors* $\mathfrak{k} = (b,c_0)$, *and let* $\mathfrak{L}$ *denote the set of operators* L *with* $(b,c_0) \in \mathfrak{K}$.

If, in the following, connected with some operator $L \in \mathfrak{L}$, an operator L_1 is considered, this operator L_1 shall be defined by (5.8) through (5.12) with the coefficients (b,c_0) of L. For example, in (5.15) this operator L_1 is to be used.

THEOREM 7a: *Suppose that the following assumptions are satisfied.*

1a) For each $L \in \mathfrak{L}$, *the homogeneous problem, corresponding to* (5.6), (5.7), *has no nontrivial solution.*

1b) For each $L \in \mathfrak{L}$, *the corresponding problem* (5.10) *has a unique solution, such that* (5.11) *holds.*

2. *For at least one* $L \in \mathfrak{L}$, *the property* (5.15) *holds.*

Then, the property (5.15) *holds for all* $L \in \mathfrak{L}$.

PROOF: Suppose (5.15) does not hold for $L^1 \in \mathfrak{L}$ with coefficient vector $\mathfrak{f}^1$, but does hold for $L^o \in \mathfrak{L}$ with $\mathfrak{f}^o$. Let then $\mathfrak{f}_t \in \mathfrak{R}$ $(0 \leq t \leq 1)$ be a continuous curve which connects $\mathfrak{f}_0$ and $\mathfrak{f}_1$, and let $p_t(x)$, γ_t be the corresponding quantities defined by (5.9) through (5.12). Moreover, define $z_t \in R$ to be the solution of $L[z](x) \equiv 1$ $(0 \leq x \leq 1)$ where L has the coefficient vector $\mathfrak{f}_t$. The functions $p_t(x)$, $z_t(x)$, and γ_t depend continuously on t.

Because of Theorem 6, it is sufficient to show that the functions $z_t(x)$ satisfy $z_t \geq 0$ for $0 \leq x \leq 1$ in the sense of (5.5). We will even show that $z_t \succ 0$ for $0 \leq t \leq 1$. Notice that the inequalities on the right-hand side of (5.5), respectively (5.16), partly depend on L_1. Of course, in the definition of $z_t \geq 0$, respectively $z_t \succ 0$, the corresponding operator L_1 constructed with $\mathfrak{f}(t)$ must be used.

We have $z_0 \geq 0$ because (5.15) holds for $L = L^o$. If $z_t \succ 0$ $(0 \leq t \leq 1)$ is not true, then, because of continuity conditions, there is a smallest $t_0 \in [0,1]$ such that $z_{t_0} \geq 0$, but not $z_{t_0} \succ 0$. Such a number t_0, however, cannot exist because the operator M, corresponding to t_0, satisfies Assumption I of Theorem 1, so that $z_{t_0} \geq 0$ implies $z_{t_0} \succ 0$.

We can extend the result of Theorem 7a to operators with variable coefficient $c(x)$. Suppose that the assumptions of Theorem 7a are satisfied.

THEOREM 7b: *Let* b *and* c_0 *denote fixed numbers such that* $(b,c_0) \in \mathfrak{R}$. *If the homogeneous problem, corresponding to* (5.6), (5.7), *has no nontrivial solution for all* $(b,c(x))$ *with*

$$C(x) < c(x) \leq c_0 \quad (0 \leq x \leq 1), \tag{5.19}$$

where $C(x)$ *denotes some fixed function, then for all operators* L *with such coefficients the property* (5.15) *holds.*

The proof is similar to the proof of Theorem 5. Notice, that now L_1 is a fixed operator.

We will now give a more explicit example. Let, for $-10\pi^2 \leq b < \infty$, $\lambda = \Lambda^*(b)$ denote the smallest eigenvalue of the eigenvalue problem

$$\begin{aligned} &g^{IV} - 2bg'' + b^2 g = 4\lambda g, \quad g(x) = -g(1-x) \quad (0 \leq x \leq 1), \\ &g(0) = g'(0) = g(1) = g'(1) = 0, \end{aligned} \tag{5.20}$$

and suppose that $\lambda = \Lambda_1(b)$ is the largest eigenvalue of the problem

$$\begin{aligned} &u^{IV} - bu'' = -\lambda u \quad (0 \leq x \leq 1), \\ &u(0) = u'(0) = u(1) = u'(1) = 0. \end{aligned} \tag{5.21}$$

Then, the following Corollary is a consequence of Theorem 7 (see Figure 2).

COROLLARY: *If*

$$-10\pi^2 < b < \infty \quad \textit{and} \quad \Lambda_1(b) < c(x) \le \Lambda^*(b) \quad (0 \le x \le 1), \tag{5.22}$$

then, for all $u \in C_4[0,1]$,

$$\left.\begin{aligned} &L[u](x) \ge 0 \quad (0 \le x \le 1), \\ &u(0) = u'(0) = u(1) = u'(1) = 0 \end{aligned}\right\} \quad \textit{imply} \quad u(x) \ge 0 \ (0 \le x \le 1). \tag{5.23}$$

PROOF: Let $\mathfrak{K}$ denote the connected set of vectors $\mathfrak{k} = (b, c_0)$ with

$$-10\pi^2 < b < \infty, \quad \Lambda_1(b) < c_0, \ 0 \le c_0 < \Lambda^*(b),$$

and let $\mathfrak{L}$ be the set of the corresponding operators L. We will show that the assumptions of Theorem 7a are satisfied for this set $\mathfrak{L}$.

Because for each $L \in \mathfrak{L}$, $\Lambda_1(b) < c_0$, the homogeneous problem corresponding to (5.6), (5.7) has no nontrivial solution. Thus, Assumption 1a of Theorem 7a is satisfied.

Because $c_0 < \Lambda^*(b)$ for each $L \in \mathfrak{L}$, the corresponding problem

$$g^{IV} - 2bg'' + b^2 g = 4c_0 g \quad (0 \le x \le 1),$$
$$g(0) = g(1) = 0, \quad g'(0) = g'(1) = 1$$

has a unique solution g which, obviously, satisfies $g(x) = -g(1-x)$. Therefore, in order to prove that Assumption 1b is satisfied, we have only to show that

$$g(x) > 0 \qquad (0 < x < \tfrac{1}{2}). \tag{5.24}$$

This inequality is true for the function $g_0 = x(1-x)(1-2x)$ which

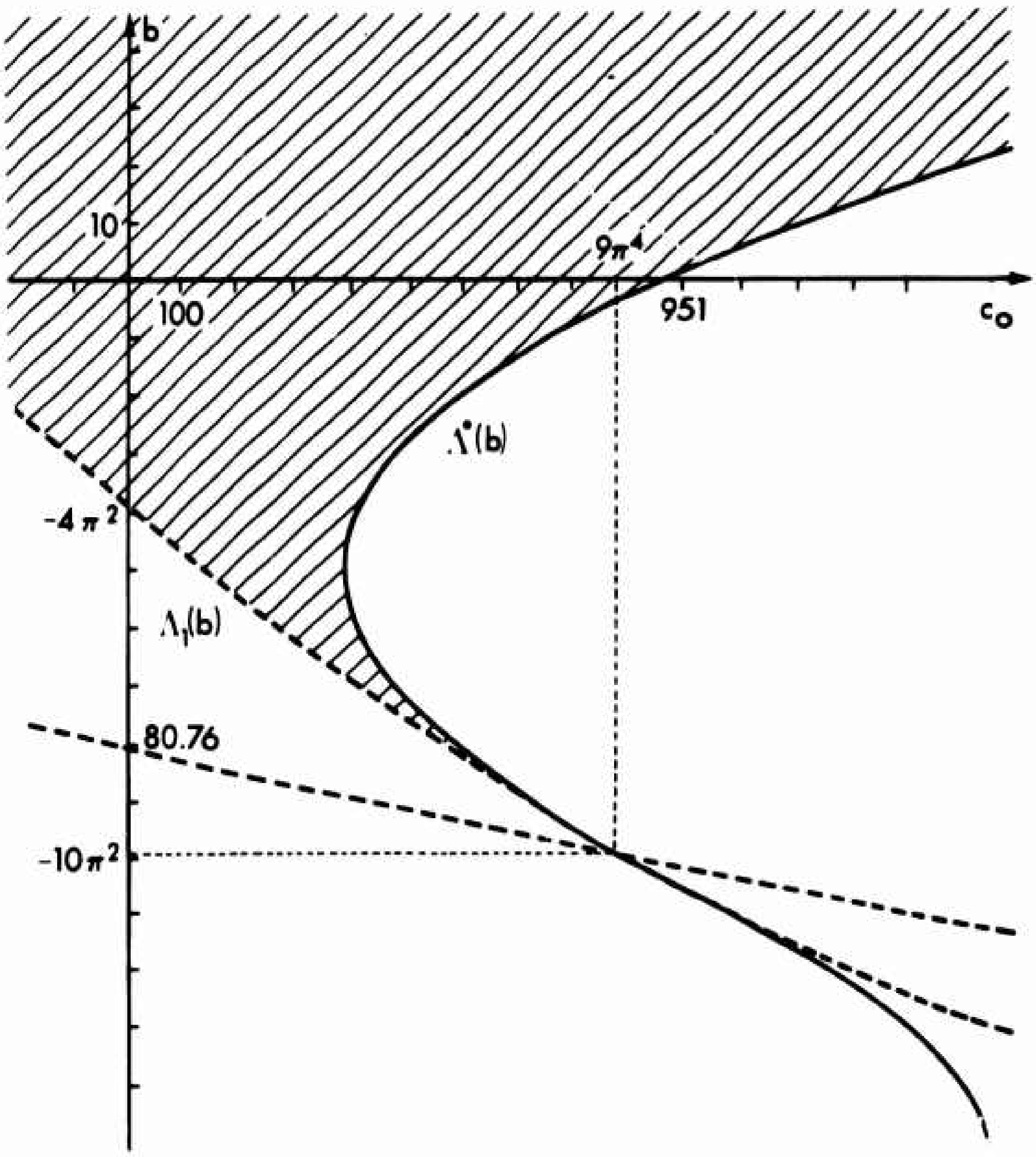

FIGURE 2:

Shaded domain: described by (5.25);

——— : eigencurve $\lambda = c_0 = \Lambda^*(b)$ of problem (5.20);

------ : eigencurves $\lambda = c_0 = \Lambda(b)$ of problem (5.21).

corresponds to $L^0[u] = u^{IV}$. Suppose, the inequality does not hold for some g^1, corresponding to an operator $L^1 \in \mathfrak{L}$. Then, by continuity arguments of a similar kind as we have used before, one can conclude that there exists a function g associated with an operator $L \in \mathfrak{L}$ such that $g(\xi) = g'(\xi) = 0$ for some $\xi \in (0,\frac{1}{2}]$.

Suppose first, that this operator L has coefficients b,c_0 satisfying $b^2 < 4c_0$. Then,

$$g(x) = \frac{1}{\Delta(\alpha,\beta)} [\sin\beta \cdot \sinh 2\alpha(x-\tfrac{1}{2}) - \sinh\alpha \cdot \sin 2\beta(x-\tfrac{1}{2})],$$

where $2\alpha = \sqrt{2\sqrt{c_0} + b}$, $\quad 2\beta = \sqrt{2\sqrt{c_0} - b}$, and

$\Delta(\alpha,\beta) = 2\alpha \sin\beta\cdot\cosh\alpha - 2\beta \sinh\alpha\cdot\cos\beta$.

The value $\Delta(\alpha,\beta)$ does not vanish for any numbers α,β which belong to a vector $(b,c_0) \in \mathfrak{K}$, because $\Delta(\alpha,\beta) = 0$ determines the eigenvalues of the problem (5.20), and $c_0 < \Lambda^*(b)$.

The argument ξ, mentioned above, cannot be $\xi = \frac{1}{2}$ because $g'(\frac{1}{2}) = 0$ would yield $\sinh\alpha/\alpha = \sin\beta/\beta$ which is not satisfied for any $\alpha > 0$, $\beta > 0$.

For $0 < \xi < \frac{1}{2}$, we calculate

$$0 = \Delta(\alpha,\beta)[g(\xi)\cdot 2\alpha\cdot\cosh 2\alpha(\xi-\tfrac{1}{2}) - g'(\xi)\cdot\sinh 2\alpha(\xi-\tfrac{1}{2})]$$

$$= -(\xi-\tfrac{1}{2})^{-1}\cdot\tfrac{1}{2}\sinh\alpha\cdot\Delta(\tilde{\alpha},\tilde{\beta})$$

with $\tilde{\alpha} = t\alpha$, $\tilde{\beta} = t\beta$, and $0 < t = 2(\xi-\frac{1}{2}) < 1$. However, $\Delta(\tilde{\alpha},\tilde{\beta}) \neq 0$

because the vector $(\tilde{b},\tilde{c}_0)$ corresponding to $\tilde{\alpha},\tilde{\beta}$, also belongs to $\mathfrak{K}$.

This proves (5.24) for all $L \in \mathfrak{K}$ with $b^2 < 4c_0$. The other cases, where $b^2 \geq 4c_0$, can be treated similarly.

Assumption 2 of Theorem 7a is also satisfied, because, as we have seen at the end of Section 5.2, the property (5.15) holds for the operator $L[u] = u^{IV}$ which belongs to $\mathfrak{K}$.

In order to show that (5.15) holds for all $u \in R$ if

$$-10\pi^2 < b < \infty \quad \text{and} \quad \Lambda_1(b) < c(x) \leq \Lambda^*(b) - \delta(b) \quad (0 \leq x \leq 1) \quad (5.25)$$

with some sufficiently small $\delta(b) > 0$, we apply Theorem 7b with $C(x) = \Lambda_1(b)$, and $c_0 = \Lambda^*(b) - \delta(b)$ for each fixed $b \in (-10\pi^2, \infty)$. It has only to be proved that the equation $u^{IV} - bu'' + c(x)u \equiv 0$ yields $u(x) \equiv 0$ for $u \in R$. This can be done by a standard procedure. If $u(x) \not\equiv 0$, then

$$0 = \int_0^1 u(u^{IV} - bu'' + c(x)u)dx = \int_0^1 (u''u'' + bu'u' + c(x)u^2)dx$$

$$> \int_0^1 (u''u'' + bu'u')dx + \Lambda_1(b)\int_0^1 u^2(x)dx.$$

This is a contradiction to the fact that $\Lambda_1(b)$ is the largest eigenvalue of the problem (5.21).

Thus, under the assumption (5.25), the implications (5.15) and (5.23) hold for all $u \in R$. In order to show that (5.23) is even true for all $u \in C_4[0,1]$ if the weaker assumption (5.22) is satisfied, one uses the fact that the Green's function, corresponding to the problem (5.6), (5.7) exists for $c(x) > \Lambda_1(b)$ and depends continuously on

$c(x)$, and that this function is non-negative if and only if (5.23) is true for all $u \in R$, or all $u \in C_4[0,1]$.

6. NUMERICAL APPLICATION

6.1 A METHOD OF APPROXIMATION AND ESTIMATION

Let there be given a differential equation

$$L[u](x) = s(x) \qquad (0 \leq x \leq 1)$$

of the N-th order (for example, $N = 2$, or $N = 4$), together with appropriate linear (inhomogeneous) boundary conditions, and suppose that

$$L[u](x) \geq 0 \quad (0 \leq x \leq 1) \quad \text{implies} \quad u(x) \geq 0 \quad (0 \leq x \leq 1)$$

for all $u \in C_N[0,1]$ which satisfy the corresponding homogeneous boundary conditions. Assume, moreover, that the given problem has a solution $u^* \in C_N[0,1]$.

Then, one may calculate an approximation ϕ for u^* and bounds for the error $u^* - \phi$ in the following way.

STEP A (Approximation). A development

$$\phi = \phi_0 + \alpha_1\phi_1 + \dots + \alpha_m\varphi_m$$

is set up such that for arbitrary constants $\alpha_1,\dots,\alpha_m$, the function ϕ is contained in $C_N[0,1]$ and satisfies the given boundary conditions.

The constants α_i are determined such that the *defect*

$$d[\phi](x) = -L[\phi](x) + s(x)$$

is orthogonal to m appropriately chosen functions ψ_k $(k = 1,2,\dots,m)$, with respect to an inner product

$$(u,v) = \int_0^1 w(x)u(x)v(x)dx, \tag{6.1}$$

where $w \varepsilon C_0[0,1]$ denotes some weight function. That means, the constants α_i constitute a solution of the linear system

$$\sum_{i=1}^{m} g_{ik}\,\alpha_i + r_k = 0 \quad (k = 1,2,\ldots,m) \tag{6.2}$$

with

$$g_{ik} = (\omega_i,\psi_k), \quad \omega_i = -L[\phi_i] \quad (i,k = 1,2,\ldots,m),$$

$$r_k = (\omega_0,\psi_k), \quad \omega_0 = d[\phi_0] \quad (k = 1,2,\ldots,m).$$

STEP E (Estimation). A function $z \varepsilon C_N[0,1]$ satisfying the homogeneous boundary conditions is chosen, and a constant $\lambda \geq 0$ is calculated such that

$$|d[\phi](x)| \leq \lambda L[z](x) \quad (0 \leq x \leq 1). \tag{6.3}$$

Then, the error estimation

$$|u^*(x) - \phi(x)| \leq \lambda z(x) \quad (0 \leq x \leq 1) \tag{6.4}$$

holds.

This procedure has been programmed for a computer. In the program, the occurring integrals (6.1) are calculated by some approximation method, and the system (6.2) is solved by an elimination procedure. In order to obtain a number λ satisfying (6.3), the function $d[\phi](x)/L[z](x)$ is calculated, and the corresponding graph is dotted for a large number of points x.

6.2 AN EXAMPLE OF THE SECOND ORDER

The boundary value problem

$$-u'' + 2xu' + (1-x^2)u = 1-x^2 \quad (0 \leq x \leq 1),$$
$$u'(0) = u(1) = 0$$

has the exact solution

$$u^* = 1 - \exp\tfrac{1}{2}(x^2-1).$$

We calculated approximate solutions with

$$\phi_0 \equiv 0, \quad \phi_i = 1-x^{i+1} \quad (i = 1,2,\ldots,m),$$

but chose w and the ψ_k in three different ways (Cases 1,2,3). For the error estimation, the function $z = z^0$ with

$$z^0(x) = \tfrac{1}{2}(e - e^{x^2}) \quad \text{and} \quad L[z^0] = e^{x^2} + (1-x^2)z^0 \qquad (6.5)$$

was used. The Tables 1,2,3 contain some of the corresponding results, namely

$$\delta = ||u^* - \phi||, \quad \text{and} \quad \lambda||z|| = \tfrac{1}{2}\lambda e \quad \text{with} \quad \lambda = ||d[\phi]/L[z]||,$$

where the notation

$$||u|| = \max\{|u(x)|: 0 \leq x \leq 1\}$$

is used. Obviously, $\lambda||z||$ is the maximum of the bound for the error $|u^* - \phi|$. The numbers δ are rounded in the usual way, the numbers $\lambda||z||$ are rounded up to the next larger decimal.

The simplest weight function $w(x) \equiv 1$, and the functions

$\psi_k = x^{k-1}$ $(k = 1,2,\ldots,m)$ give the results in Table 1, for $m = 2,3,4,5$ (Case 1). For this case, the maximal values of the quotient $|d[\phi](x)|/L[z](x)$ occur near the boundary points $x = 0$ and $x = 1$.

Therefore, in the second case, the same functions ψ_k, but weight functions

$$w(x) = [1 - \gamma(2x-1)^2]^{-1} \qquad (\gamma = \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{7}{8})$$

are used which have larger values near the boundary points than in the middle of the interval. The corresponding results for $m = 4$ are contained in Table 2.

A similar effect can be obtained with the simple weight function $w(x) \equiv 1$, but rational functions ψ_k. For $m = 4$,

$$\psi_1 \equiv 1, \qquad \psi_2 = x, \qquad \psi_3 = \frac{1}{x - c} \qquad \psi_4 = \frac{1}{x - (1+c)},$$

and different values of c, one gets the results in Table 3.

The result in the second case, for $\gamma = 7/8$, comes close to that which one would gain by requiring that the Tschebyscheff norm of $d[\phi](x)/L[z](x)$ is minimal, because the extrema of this quotient have alternate signs and almost equal absolute values.

In all cases which we treated, we obviously did not get very much smaller error bounds by using another approach than the simplest one in Case 1.

TABLE 1 $(w(x) \equiv 1,\ \psi_k = x^{k-1})$

m	$\delta = \|\|u^* - \phi\|\|$	$\lambda\|\|z\|\| = \lambda\cdot\frac{1}{2}e = \|\|$ error bound $\|\|$
3	0.000 715	0.033
4	0.000 069 0	0.008 3
5	0.000 005 04	0.000 92
6	0.000 000 525	0.000 15

TABLE 2 $(w = [1 - \gamma(2x - 1)^2]^{-1},\ \psi_k = x^{k-1},\ m = 4)$

γ	$\delta = \|\|u^* - \phi\|\|$	$\lambda\|\|z\|\| = \lambda\frac{1}{2}e = \|\|$ error bound $\|\|$
0	0.000 069	0.008 3
0.5	0.000 114	0.007 3
0.75	0.000 157	0.006 3
0.875	0.000 192	0.005 4

TABLE 3 $(w \equiv 1,\ m = 4,\ \psi_k$ as in (6.6))

c	$\delta = \|\|u^* - \phi\|\|$	$\lambda\|\|z\|\| = \lambda\frac{1}{2}e = \|\|$ error bound $\|\|$
1	0.000 077	0.008 1
0.5	0.000 088	0.007 9
0.2	0.000 116	0.007 3
0.01	0.000 256	0.006 3

It was not necessary to use orthogonalized functions ψ_k in order to avoid undesired consequences of rounding errors in solving the linear system. (For corresponding error estimations for partial differential equations, we had to use orthogonalized functions ψ_k.)

6.3 CONCERNING THE ACCURACY OF THE ERROR ESTIMATION

For the examples of Section 6.2, the actual error $|u^* - \phi|$ is much smaller than the error bound, and the ratio δ/λ decreases with increasing m. This has to be expected in general, if one calculates error bounds for different m in the same simple way, as we did.

If one requires that the defect is orthogonal to all polynomials up to the degree $m - 1$, or to functions with similar behavior, the defect assumes the value 0 at least m times in the interval $[0,1]$. The error $u^* - \phi$ then usually oscillates similarly. However, because the defect $d[\phi] = L[u^* - \phi]$ contains derivatives of the error, this defect will not decrease as fast as the error, with increasing m. For example, if $L[u] = -u''$, and $u^* - \phi^m = f(m) \sin(m-1)\pi x$ with some (decreasing) function $f(m)$, then $d[\phi^m] = (m-1)^2\pi^2(u^*-\phi^m)$.

If one wants to get an error bound which is close to the actual error, this bound need have a similar oscillatory behavior as $u^* - \phi$. To calculate such a bound requires much more work than to get such simple bounds as in Section 6.2. This additional effort is better used to obtain better approximations.

For example, let $\phi = \phi^m$ be the approximations of Case 1 for different $m = 3,4,5,6$, and denote by λ_m the corresponding value λ in Table 1. Then, for $\phi = \phi^3$, the inequality (6.3) is satisfied for $\lambda = 1$ and $z = \zeta$ with $\zeta = \phi^3 - \phi^5 + \lambda_5 z^0$ and z^0 as in (6.5). This function ζ is a bound for $|u^* - \phi^3|$ which is not much larger than $|u^* - \phi^3|$ itself, and the similar function $\phi^3 - \phi^6 + \lambda_6 z^0$ is an even better bound. But, it is not reasonable to use the estimation $|u^* - \phi^3| \leq \zeta$. It would be better to use $|u^* - \phi^5| \leq \lambda_5 z^0$.

The question of practical interest is not, how close to the actual error the error bound is, but how small a bound one can get with a certain amount of work.

If one wants to get exact error bounds, one has to take into account the rounding errors. The rounding errors which occur during the calculation of the constants α_i might influence the accuracy of the approximation ϕ, but they do not destroy the validity of the corresponding error estimation. Only the rounding errors which occur when ϕ is calculated with the computed α_i, and those which occur during Step E have to be considered in some way. In particular, one has to be careful in calculating the defect $d[\phi]$ which usually is a small number computed as a difference of larger numbers. Eventually, one has to compute the defect with double precision. This was not necessary for our examples.

6.4 EXAMPLES OF THE FOURTH ORDER

We treated the boundary value problem

$$u^{IV} - bu'' + \mu[1 + x(1 - x)]u = 1000(1 + x) \qquad (0 \leq x \leq 1)$$

$$u(0) = u'(0) = u(1) = u'(1) = 0$$

for different constants b and μ. Approximate solutions ϕ have been calculated using

$$\phi_0(x) \equiv 0, \quad \phi_i = x^2(1 - x)^2 x^{i-1} \qquad (i = 1,2,\ldots,m),$$

$$w(x) \equiv 1, \quad \psi_i = x^{i-1} \qquad (i = 1,2,\ldots,m).$$

The method of error estimation in Section 6.2 can be applied if (5.22) is satisfied for $c(x) = \mu[1 + x(1-x)]$, i.e. if

$$-10\pi^2 < b < \infty, \quad \tfrac{4}{5}\Lambda_1(b) < \mu \leq \tfrac{4}{5}\Lambda^*(b) \tag{6.6}$$

If b and μ are small enough, one can use

$$z = x^2(1 - x)^2 \tag{6.7}$$

because then $L[z](x) > 0$. This was done in

CASE 1: $b = 0$; $\mu = 1$.

The Table 4 contains some of the corresponding results:

$$\phi(\tfrac{1}{2}), \quad ||d[\phi]||, \quad \text{and} \quad \lambda||z|| \quad \text{with} \quad \lambda = ||d[\phi]/L[z]||. \tag{6.8}$$

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The values $\phi(\frac{1}{2})$ are listed with all decimals that were calculated by the computer. Therefore, the last decimals might be influenced by rounding errors.

For

CASE 2: $b = 6, \quad \mu = 100,$

different functions z have been used, namely z in (6.7), and the solutions $z = z_k$ of the problem

$$z^{IV} - bz'' + k\cdot\mu z = 1, \quad z(0) = z'(0) = z(1) = z'(1) = 0$$

for

$$k = 1, 1.25, 2, 4, 10. \tag{6.9}$$

Except the cases $m = 4, k = 8$; $m = 4, k = 10$; and $m = 7, k = 10$, the maximum of $d[\phi]/L[z]$ occurred at one of the boundary points, so that then

$$||\text{error bound}|| = \lambda||z|| = \frac{||d[\phi]||}{|L[z](0)|}||z|| = \nu||d[\phi]||, \tag{6.10}$$

where the numbers ν are listed for different cases in Table 5A.

The Table 5B contains the quantities (6.8) for Case 2 with $z = z_{10}$.

For

CASE 3: $b = 20; \quad \mu = 200, 400, 800, 1200,$

we have checked how the approximation and estimation procedure behaves with increasing $c(x)$. For $b = 20$, the second relation in (6.6) becomes

$$595 \approx \frac{4}{5}\Lambda_1(b) < \mu \leq \frac{4}{5}\Lambda^*(b) \approx 1229$$

Among the functions z_k with k as in (6.9), z_4 gives the smallest bounds for most of the numbers m which we used. Table 6 contains some of the corresponding results.

Finally, in

CASE 4: $b = 20$; $\mu = -200, -400, -500,$

we checked how the error bounds behave with decreasing $c(x)$. Some results are given in Table 7. For simplicity, we used $z = z_1$ although other functions z_k might give better bounds.

TABLE 4 ($b = 0$, $\mu = 1$)

m	$\phi(\frac{1}{2})$	$\lvert\lvert d[\phi] \rvert\rvert$	$\lambda \lvert\lvert z \rvert\rvert = \lvert\lvert$error bound$\rvert\rvert$ for z in (6.7)
2	3.899 673 1	2.595	0.006 8
3	3.896 778 0	1.258	0.003 3
4	3.896 778 4	1.232	0.003 3
5	3.896 572 6	0.117	0.000 31

TABLE 5A ($b = 6$, $\mu = 100$)

z		ν in (6.10)
z as in (6.7)		0.005 209
z_k with $k =$	1	0.001 925
	1.25	0.001 855
	2	0.001 673
	4	0.001 325
	10	0.000 809

TABLE 5B ($b = 6$, $\mu = 100$)

m	$\phi(\frac{1}{2})$	$\lvert\lvert d[\phi] \rvert\rvert$	$\lambda \lvert\lvert z \rvert\rvert = \lvert\lvert$error bound$\rvert\rvert$ for $z = z_{10}$
3	2.797 274 9	104.98	0.085
4	2.797 275 1	51.27	0.057
5	2.791 040 6	4.67	0.003 8
6	2.791 041 5	2.47	0.002 0
7	2.790 992 4	1.32	0.001 4

TABLE 6 (b = 20)

μ	m	$\phi(\frac{1}{2})$	$\|\|d[\phi]\|\|$	$\lambda\|\|z\|\| = \|\|$error bound$\|\|$ for $z = z_4$
400	3	1.551 691 4	241.09	0.13
	4	1.551 691 4	57.51	0.032
	5	1.556 082 7	0.904	0.000 67
	6	1.556 083 9	0.602	0.000 66
800	3	1.102 873 6	214.90	0.083
	4	1.102 873 6	22.61	0.071
	5	1.102 733 9	20.33	0.064
	6	1.102 733 9	12.74	0.049
	7	1.102 889 5	1.40	0.000 44
1200	3	0.851 844 09	262.68	0.076
	4	0.851 844 12	63.02	0.014
	5	0.850 614 11	39.51	0.008 6
	6	0.850 614 12	25.55	0.007 1
	7	0.850 904 94	2.54	0.000 55

TABLE 7 (b = 20)

μ	m	$\phi(\frac{1}{2})$	$\|\|d[\phi]\|\|$	$\lambda\|\|z\|\| = \|\|\text{error bound}\|\|$ for $z = z_1$
	3	3.841 732 4	636.30	1.6
	4	3.841 732 5	495.95	1.2
-200	5	3.914 621 2	31.14	0.075
	6	3.914 623 0	14.83	0.036
	7	3.914 505 0	6.07	0.015
	3	7.423 625 7	1 317	5.0
	4	7.423 626 5	1 193	4.6
-400	5	7.755 458 4	33.20	0.13
	6	7.755 471 1	9.84	0.038
	7	7.755 355 1	2.05	0.0078
	3	13.818 191	2 554	16
	4	13.818 193	2 439	16
-500	5	15.109 652	37.03	0.20
	6	15.109 649	10.26	0.099
	7	15.109 907	4.47	0.024

7. REFERENCES

For differential equations of the first and second order, monotonic properties have been used for a long time (see, for example, [5], [2]). Chaplygin studied extensively the application of such properties for numerical calculations. Since then, a large number of papers on that subject have been published. We list some books and papers where corresponding references and numerical examples can be found. Besides that, only a few papers are listed that are directly referred to in this article.

[1] Beckenbach, E. F., and Bellman, R.: Inequalities. Berlin-Göttingen-Heidelberg, 1961, pp. 148 ff.

[2] Chaplygin, S. A.: New Methods in the Approximate Integration of Differential Equations (Russian). Moscow, Gosudarstvd. Izdat. Tech.-Teoret. Lit. 1948.

[3] Collatz, L.: Aufgaben monotoner Art. Arch. Math. 3, 365-376 (1952).

[4] Collatz, L.: Funktionalanalysis und Numerische Mathematik. Berlin-Göttingen-Heidelberg (1964), pp. 300 f.

[5] Perron, O.: Ein neuer Existenzbeweis für die Integrale der Differentialgleichung $y' = f(x,y)$. Math. Ann. 76, 471-484, (1915).

[6] Schröder, J.: Lineare Operatoren mit positiver Inversen. Arch. Rational Mech. Anal. 8 (1961), 408-434.

[7] Schröder, J.: Monotonie-Eigenschaffen bei Differentialgleichungen. Arch. Rational Mech. Anal. 14 (1963) 38-60.

[8] Schröder, J.: Estimations in Nonlinear Equations Proceedings, IPIP-Congress 1965, In print.

[9] Schröder, J.: Randwertaufgaben Vierter Ordnung mit positiver Greenscher Funktion. Math. Z. In print.

[10] Walter, W.: Differential-und Integral-Ungleichungen. Berlin-Göttingen-Heidelberg-New York (1964).